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Spin-up of a rapidly rotating heavy gas in a thermally insulated annulus

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The linear spin-up problem for a rapidly rotating viscous diffusive ideal gas is considered in the limit of vanishing Ekman number E. Particular attention is given to gases having a large molecular weight. The gas is enclosed in a cylindrical annulus, with flat top and bottom walls, which is rotating around its axis of symmetry with rotation rate Ω. The walls of the container are adiabatic. In a rotating gas (of any molecular weight), the Ekman layers on adiabatic walls are weak, which implies that there is no distinct non-diffusive response of the gas outside the Ekman and Stewartson boundary layers on the timescale $E^{-1/2}\Omega^{-1}$ for spin-up of a homogeneous fluid. For the case of adiabatic walls, it is shown that the spin-up mechanisms due to viscous diffusion and Ekman suction are, from a formal point of view, equally strong. Therefore, the gas will adjust to the increased rotation rate of the container on the diffusive timescale $E^{-1}\Omega^{-1}$. However, if $E^{1/3} \ll \gamma - 1 \ll 1$ and $M \sim 1$, which characterizes rapidly rotating heavy gases (where γ is the ratio of specific heats of the gas and M the Mach number), it is shown that the gas spins up mainly by Ekman suction on the shorter timescale $(\gamma-1)^2 E^{-1}\Omega^{-1}$. In such cases, the interior motion splits up into a non-diffusive part of geostrophic character and diffusive boundary layers of thickness $(\gamma-1)$ outside the Ekman and Stewartson layers. The motion approaches the steady state of rigid rotation algebraically instead of exponentially as is usually the case for spin-up.

1. Introduction

Many theoretical studies have been devoted to various aspects of rapidly rotating gas flows. Although the present paper is mainly academic, there are engineering applications of the theory of rapidly rotating gases. The most important application is centrifugal enrichment of uranium-hexafluoride, which is a gas of large molecular weight. Another interesting related application that involves flow of air is the control of weather in space colonies (Matsuda 1982). In this case, an artificial gravitational field is assumed to be set up by rotation, which, among other things, will lead to Coriolis effects on the atmosphere in the space colony. Although air is not a heavy gas, some of the phenomena discussed in the present paper are certainly also present in that application. The first papers treating rapidly rotating gases from a fundamental fluid mechanics point of view appear to have been published by Japanese researchers, notable examples being the work by Mikami (1973), Nakayama & Usui (1974) and Sakurai & Matsuda (1974). Illuminating review articles on the gasdynamics of rapidly rotating centrifuges have been written by Rätz (1978) and Soubbaramayer (1979). An account of later developments of the subject can be found in Takashima (1986).

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Most published studies of rapidly rotating gas flows deal with steady flows. Some of the few exceptions are the papers by Morton & Shaughnessy (1972), Gans (1974), Hultgren (1978) and Miles (1981) in which adiabatic and inviscid oscillations were investigated. The linear spin-up problem for a rapidly rotating gas in a thermally conducting container has been treated by Bark, Meijer & Cohen (1979). In that study all oscillatory motions were filtered out by consideration of the long-time response of the gas to a small impulsive change of the rotation rate of the container. Recently, numerical solutions of the complete problem for spin-up from rest of a rapidly rotating gas in a container with conducting walls have been given in the literature, see Hyun & Park (1989, 1992). In the present paper, it is shown that the linear spin-up of a rapidly rotating gas in a container with adiabatic walls is completely different from the case with thermally conducting walls.

The basic work on a linear spin-up of a homogeneous fluid is the paper by Greenspan & Howard (1963), who computed an exact integral transform solution for a fluid between two parallel infinite plates and an asymptotic solution for closed axisymmetric containers. Those authors showed that if the rotation rate of the solid walls is instantaneously changed from Ω to $\Omega+\Delta\Omega$, the fluid spins up on the timescale $E^{-1/2}\Omega^{-1}$, where E is the Ekman number, rather than on the viscous diffusion timescale $E^{-1}\Omega^{-1}$. The reason for this somewhat surprising result is that the main body of fluid is spun up inviscidly due to suction of fluid into the viscous Ekman layers on non-vertical walls. As the fluid is incompressible, the motion into the Ekman layers is accompanied by a radial motion and the adjustment to the new state of rigid rotation turns out to be a consequence of conservation of angular momentum. In addition to the motion that is 'driven' by the Ekman layers, weak oscillations appear, whose frequency spectrum is confined to the range $(-2\Omega, 2\Omega)$. These oscillations, which decay on the timescale $E^{-1}\Omega^{-1}$, have no effect on the spin-up process. The theoretical results were found to agree very well with experiments. Later contributions to the study of this phenomenon are discussed in the review by Benton & Clark (1974).

Spin-up motion occurs in several geophysical and astrophysical phenomena, and there are numerous studies in the literature of the spin-up of a contained axially stratified Boussinesq fluid. Contradictory theoretical results were obtained by Holton (1965) and Pedlosky (1967), who both made a number of *ad hoc* assumptions about the structure of the boundary layers. The disagreement was settled, in favour of Holton, by the work of Sakurai, Clark & Andre Clark (1969), Walin (1969) and Sakurai (1969). These authors showed that the response on the timescale $E^{-1/2}\Omega^{-1}$ of an axially stratified Boussinesq fluid is qualitatively similar to that of a homogeneous fluid. However, the axial stratification tends to block the Ekman suction mechanism and the fluid does not spin up to a state of rigid rotation on this timescale. Instead, a quasi-steady state is reached, which slowly evolves towards rigid rotation on the diffusive timescale $E^{-1}\Omega^{-1}$. The multiple-scale character of the spin-up of an axially stratified Boussinesq fluid has been clarified by St-Maurice & Veronis (1975). Later developments in this field were also discussed by Benton & Clark (1974).

For a rapidly rotating gas in an axisymmetric container with thermally conducting walls, there is also an intermediate quasi-steady state of non-rigid rotation that is asymptotically reached by the gas on the $E^{-1/2}\Omega^{-1}$ timescale due to the Ekman suction mechanism (Bark *et al.* 1979). As in the case of a Boussinesq fluid, this state is gradually modified by diffusive effects and possibly weak Ekman suction until the gas is rotating rigidly with the same angular velocity as the container. However, if the walls of the container are thermally insulating, which is the problem to be considered in the present paper, the character of the motion will change significantly. The reason is that, in linear flows of rapidly rotating gases, Ekman layers on thermally insulating walls are weak. This has been shown by Bark & Hultgren (1979) for steady motions but the result carries over to motion on the $E^{-1/2}\Omega^{-1}$ and $E^{-1}\Omega^{-1}$ timescales. It was shown by those authors that the flow in an Ekman layer on a thermally insulated surface is a factor $E^{1/2}$ weaker than on a thermally conducting surface. As a consequence, the quasi-steady intermediate state that is reached on the $E^{-1/2}\Omega^{-1}$ timescale if the walls are conducting disappears if the walls are insulating. This means that the timescale on which the Ekman suction affects the gas outside the Ekman layers is the diffusive timescale.

When γ, the ratio of the specific heats at constant pressure and volume, respectively, is only slightly larger than unity, i.e. when the gas under consideration is heavy, it is shown in the present paper that the Ekman suction becomes relatively more important in a container with adiabatic walls. The gas spins up on a timescale $(\gamma-1)^2 E^{-1}\Omega^{-1}$, which is short compared to the diffusive timescale but large compared to the spin-up timescale for a homogeneous fluid, under the assumption $E^{1/3} \ll \gamma-1 \ll 1$, and peripheral Mach numbers M of order unity. The main part of the interior motion is inviscid and non-diffusive but vertical and horizontal boundary layers of thickness $(\gamma-1)$ appear outside the Stewartson and Ekman layers. A simplified asymptotic analysis of these boundary layers and the spin-up problem is made by consideration of the limit $\gamma \to 1^+$ and $M = O(1)$.

In mathematical or numerical solutions of problems of spin-up in cylindrical containers, the logarithmic singularity on the axis of symmetry poses no difficulties. Unfortunately, this is not so in the problem that is dealt with in the present work, where a complicated singularity appears at zero radius. The somewhat unusual nature of this singularity, which, on the timescale considered, implies instantaneous spin-up on the axis of symmetry, is briefly discussed at the end of §2. To avoid numerical difficulties associated with this singularity, attention is in the present work restricted to annular containers. However, it should be pointed out that annular containers are common in centrifugal separation of gaseous UF_6, see e.g. Bermel, Coester & Rätz (1989).

The present paper is laid out as follows. Section 2 contains the mathematical statement of the problem, a derivation of a simplified linearized set of equations of motion and a heuristic derivation of the boundary conditions. The problem formulation in §2 is valid for a gas of any molecular weight. A numerical solution of the linearized problem is presented in §3. In §4, further simplifications are made by assuming that γ is close to one. A significant part of the details of the derivations and discussions in §4 is relegated to Appendices B and C. As a supplement to the somewhat formal development in §4 and the Appendices, §5 contains a discussion in qualitative terms of the spin-up of a heavy gas in a container, whose walls are adiabatic, with particular attention to the role of the Ekman layers. The main conclusions are summarized in §6.

2. Formulation

Consider a viscous and thermally conducting ideal gas, whose mechanical behaviour is characterized by μ, ϑ the dynamic shear and expansion viscosities; c_p, c_v, the specific heats at constant pressure and volume, $\gamma = c_p/c_v$; and κ, the thermal conductivity. These quantities are assumed to be constants. The gas is contained in a rotating concentric cylindrical annulus of height $2H$. The inner and outer diameters of the annulus are $2r_i H$ and $2r_o H$, respectively. The geometry of the container is shown in figure 1. (The case $r_i = 0$ turns out to be somewhat spurious from a mathematical point

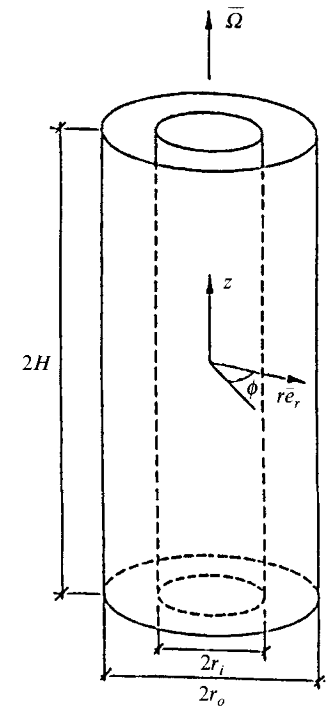


FIGURE 1. Geometry of container and definition of coordinate system.

of view and is briefly commented upon at the end of this section.) A cylindrical coordinate system (r, ϕ, z) is used. The coordinate system, which is rotating with the container, is chosen such that the flat top and bottom walls of the container are at $z = \pm H$ and the axis of symmetry, around which the container rotates, coincides with the z-axis. Initially, the temperature T of the gas is assumed to be constant $(= T_0)$ and the gas to be rotating rigidly with the angular velocity Ω of the container. Ω is assumed to be sufficiently large for effects of gravity to be negligible. The governing dimensional equations are

$$\rho[\boldsymbol{u}_t + \tfrac{1}{2}\nabla \boldsymbol{u}\cdot\boldsymbol{u} + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} - \Omega^2 r\boldsymbol{e}_r + 2\Omega\boldsymbol{e}_z\times\boldsymbol{u}]$$
$$= -\nabla p - \mu(\nabla\times\nabla\times\boldsymbol{u} - \tfrac{2}{3}\nabla\nabla\cdot\boldsymbol{u}) + \vartheta\nabla\nabla\cdot\boldsymbol{u}, \quad (2.1\,a)$$

$$\rho_t + \nabla\cdot(\rho\boldsymbol{u}) = 0, \quad (2.1\,b)$$

$$\rho T(s_t + \boldsymbol{u}\cdot\nabla s) = \kappa\nabla^2 T, \quad (2.1\,c)$$

$$p = R\rho T, \quad s = c_v \log(p\rho^{-\gamma}) + \text{const.} \quad (2.1\,d)$$

Here $\boldsymbol{u} = (u, v, w)$, p, ρ and s are the velocity, pressure, density and entropy fields, respectively, t is time and R is the gas constant. These equations are to be solved subject to the initial and boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0}, \quad T = T_0, \quad \rho = \rho_0(r) = \rho(r_0 H)\exp\left\{\frac{\gamma M^2}{2}\left[\left(\frac{r}{r_0 H}\right)^2 - 1\right]\right\} \quad \text{for} \quad t = 0, \quad (2.2)$$

$$\boldsymbol{u} = \epsilon r\times\Omega\boldsymbol{e}_z, \quad \boldsymbol{n}\cdot\nabla T = 0 \quad \text{at} \quad z = \pm H, \quad r_i H < r < r_o H$$
$$\text{and} \quad r = r_{i,o}H, \quad |z| < 1 \quad \text{for} \quad t > 0. \quad (2.3)$$

In (2.2), M is the Mach number, defined as $M = r_o H\Omega/(\gamma R T_0)^{1/2}$. The Rossby number ϵ in (2.3) is defined as $\epsilon = \Delta\Omega/\Omega$, where $\Delta\Omega > 0$ is the increase of the rotation rate of the container. It is assumed that ϵ is sufficiently small for linear theory to be valid. The following additional non-dimensional parameters will appear:

$$\text{Ekman number,}\quad E = \mu/\rho_0(r_o H)\,H^2\Omega;$$
$$\text{Prandtl number,}\quad \sigma = \mu c_p/\kappa;$$
$$\alpha^2 = \sigma(\gamma-1)\,M^2/(2r_o)^2.$$

The problem defined by (2.1 *a–d*), (2.2) and (2.3) will be considered in the limit

$$E \to 0+,\quad M = O(1),\quad \sigma = O(1),\quad \alpha^2 = O(1),\quad r_{i,o} = O(1),\quad \epsilon \ll 1 \quad \text{and fixed.}$$

In this limit, Ekman layers will appear on the endwalls of the container and Stewartson layers on the cylindrical walls. To begin with, α^2 is taken to be of order unity. The limit of vanishing α^2, i.e. for a heavy gas with γ close to 1, will be considered in §4.

It was shown by Bark & Hultgren (1979) that the role of a steady Ekman layer at an insulating wall is to correct the heat flux of the geostrophic field, and that the no-slip condition has, to lowest order, to be fulfilled by the geostrophic field itself. This was shown to imply that the strength of the Ekman-layer flow is $O(\epsilon E^{1/2})$ instead of $O(\epsilon)$ as is the case for a thermally conducting wall, see e.g. Sakurai & Matsuda (1974), and in the case of a homogeneous fluid, see e.g. Greenspan (1990). It is readily shown that the annihilation of the lowest-order Ekman-layer flow by an adiabatic wall also prevails on the timescales $E^{-1/2}\Omega^{-1}$ and $E^{-1}\Omega^{-1}$. This means that although the inviscid vortex stretching mechanism for spin-up is still present, its effect will be felt not on the $E^{-1/2}\Omega^{-1}$ timescale but on the longer $E^{-1}\Omega^{-1}$ timescale. The only relevant timescales for the spin-up process in the present case are thus Ω^{-1} and the diffusion timescale $E^{-1}\Omega^{-1}$. On the latter timescale, effects of viscous and thermal diffusion on the gas outside the boundary layers are consequently as strong as those of the weak Ekman suction. In the present work, the response of the gas will be considered on the diffusive timescale only, which means that all oscillations on the Ω^{-1} timescale are filtered out.

Non-dimensional independent variables are defined as

$$\tau = E\Omega t,\quad r^* = r/H. \tag{2.4a}$$

For the deviation from the state of rigid rotation that is specified by (2.2), non-dimensional dependent variables, which are assumed to be of order unity, are defined as

$$\left.\begin{aligned} (u^*, v^*, w^*) &= \frac{(E^{-1}u, v, E^{-1}w)}{\epsilon H\Omega},\quad p^* = \frac{p-p_0}{\epsilon\rho_0(r_0 H)\,H^2\Omega^2},\\ \rho^* &= \frac{\rho-\rho_0}{\epsilon\rho_0(r_o H)},\quad T^* = \frac{T-T_0}{\epsilon T_0},\end{aligned}\right\} \tag{2.4b}$$

where p_0 is the pressure field in the rigidly rotating gas. The function $\rho_0(r)$ is, in what follows, normalized with the value $\rho_0(r_0 H)$. Henceforth asterisks will be dropped.

Substitution of (2.4 *a, b*) into (2.1 *a–d*) gives, to lowest order in ϵ and E, the following set of equations for the geostrophic motion:

$$-2\rho_0 v - r\rho + p_r = 0,\quad \rho_0 v_\tau + 2\rho_0 u = \left(\nabla^2 - \frac{1}{r_2}\right)v, \tag{2.5a, b}$$

$$p_z = 0,\quad \rho_\tau + \frac{1}{r}(r\rho_0 u)_r + \rho_0 w_z = 0, \tag{2.5c, d}$$

$$\frac{4\alpha^2}{\gamma-1}p_\tau - \sigma\rho_\tau - 4\alpha^2\rho_0 r u = \nabla^2 T,\quad p = \frac{\sigma(\gamma-1)}{4\gamma\alpha^2}(\rho_0 T + \rho). \tag{2.5e, f}$$

Since the time derivatives of u and w, which are correction terms of $O(E)$ in (2.5*a*) and (2.5*c*), respectively, are neglected, the solution of the system (2.5*a*–*d*) cannot satisfy all initial conditions specified in (2.2). The initial conditions for u and w must therefore be dropped and the solution can only satisfy

$$v = p = \rho = T = 0 \quad \text{for} \quad \tau = 0. \tag{2.6}$$

The solution obtained in the present work is thus, as expected, invalid for τ approaching zero. One can presumably compute a uniformly valid solution by finding an approximate solution on the Ω^{-1} timescale and matching that solution as a boundary-layer solution (in time) to the solution computed in the following sections. A procedure along this line was used by St-Maurice & Veronis (1975) in their treatment of a geometrically simplified version of the spin-up problem for an axially stratified Boussinesq fluid. In the present problem, the solution on the Ω^{-1} timescale will be considerably more complicated, involving not only inertial and internal waves but also acoustic waves, see e.g. Gans (1974), and the computation of a uniformly valid solution is not attempted. However, this formal incompleteness of the solution does not appear to be very serious, neither mathematically nor physically. It is clear from the exact solution by Greenspan & Howard (1963) for a homogeneous fluid, that the only phenomenon of importance for the spin-up process taking place on the Ω^{-1} timescale is the formation of the Ekman layers. But these layers are quasi-steady on the $E^{-1/2}\Omega^{-1}$ timescale and the geostrophic motion on this timescale can be computed without knowledge of the solution on the Ω^{-1} timescale. These fortunate circumstances turn out to prevail in the case considered by St-Maurice & Veronis (1975) as well. Not surprisingly, a well-posed problem also results in the present case without consideration of the solution on the Ω^{-1} timescale, which, on reasonable grounds, thus can be assumed to be of secondary importance.

The number of boundary conditions that have to be satisfied by the solution of the simplified system (2.5*a*–*f*) is not obvious. However, this matter can be settled by examination of a reduced version of the system. Using the Ekman-suction formula derived by, among others, Bark & Hultgren (1979), i.e.

$$w(\tau, r, \pm 1) = \frac{1}{4\alpha^2 r \rho_0} T_{rz}(\tau, r, \pm 1), \tag{2.7}$$

and carrying out essentially the same algebraic manipulations that are outlined in Bark & Hultgren (1979) for the derivation of equation (3.20) in that paper, one finds, after some algebra, the following two coupled equations for the swirl velocity v and the quantity $\Xi = p/\rho_0$:

$$\left\{\Delta_r\left[\frac{\langle v\rangle}{r} - \frac{1}{2r}\Xi_r\right] - \sigma\rho_0\left[\frac{\langle v\rangle}{r} - \frac{1}{2r}\Xi_r\right]_\tau\right\}_r + 4\alpha^2\rho_0\langle v\rangle_\tau = 0, \tag{2.8a}$$

$$r\left\{\Delta_r\left[\frac{v}{r} - \frac{1}{2r}\Xi_r\right] - \sigma\rho_0\left[\frac{v}{r} - \frac{1}{2r}\Xi_r\right]_\tau\right\} + (1+\alpha^2 r^2)\,v_{zz}$$
$$+\alpha^2 r^2\left(\Delta_r - \frac{1}{r_2}\right)v - \rho_0\alpha^2 r^2\left(v - 2\frac{\Xi_r}{r}\right)_\tau = 0. \tag{2.8b}$$

In these equations, Δ_r is the radial art of the Laplacian operator in cylindrical coordinates and $\langle\ \rangle$ denotes vertical average over the container, i.e.

$$\langle v\rangle = \frac{1}{2}\int_{-1}^{1} v\,\mathrm{d}z.$$

As in the case considered by Bark & Hultgren (1979), the largest velocity component of the interior flow, i.e. the swirl velocity v, must satisfy the no-slip condition, see the first of conditions (2.3), on solid walls of the container

$$v = r \quad \text{for} \quad z = \pm 1, r_i < r < r_o \quad \text{and} \quad v = r_{i,o} \quad \text{for} \quad |z| < 1, r = r_{i,o}. \tag{2.9}$$

For any function Ξ one can express v in terms of Ξ by computing the Green's function for the parabolic differential operator in (2.8b) and the boundary conditions (2.9). If that expression for v is substituted into (2.8a), one obtains a partial integro-differential equation for Ξ of fourth order in r. One must consequently specify two further boundary conditions for Ξ at each of $r = r_i$ and $r = r_o$. It should be noted that p, and hence also Ξ, are independent of z according to (2.5c). The additional boundary conditions, the specification of which requires some discussion of the Stewartson layers, must consequently not depend on z.

It will first be shown, using essentially the same arguments as in the work by Bark & Hultgren (1979) on the steady case, that the vertically average of the heat flux associated with the geostrophic flow, i.e. $\langle T_r \rangle$, must vanish at $r = r_{i,o}$. If the stream function for the meridional flow in a Stewartson $E^{1/3}$ layer is denoted $E^{2/3}\varphi$, where φ is assumed to be of order unity, the following formula relating φ and the interior temperature field T at a thermally insulated vertical boundary can be derived (Bark & Hultgren 1979, p. 112):

$$T_r - 4\alpha^2 r_{i,o} \int_0^\infty \varphi_z \, d\xi = 0, \quad \xi = \pm \frac{r - r_{i,o}}{E^{1/3}}, \tag{2.10}$$

where ξ is the stretched radial coordinate. For a Stewartson $E^{1/3}$ layer of the assumed strength one must prescribe that $\varphi = 0$ at $z = \pm 1$ since otherwise, the axial velocity in the layer, i.e. φ_ξ, must be corrected at $z = \pm 1$ by the axial velocity of an Ekman-layer extension, whose horizontal flow is of order $E^{-1/6}$. However, there are no further correction fields available to cancel for example the heat flux of order $E^{-2/3}$ in such an Ekman layer at $z = \pm 1$. Using the fact that φ thus must vanish on the horizontal boundaries and computing $\langle T_r \rangle$ from (2.10) one finds that a non-zero value of $\langle T_r \rangle$ cannot be corrected at a vertical boundary by a Stewartson $E^{1/3}$ layer. The only remaining possibility to correct this quantity to zero at $r = r_{i,o}$ would be to introduce a Stewartson $E^{1/4}$ layer, whose swirl velocity is of order $E^{1/4}$. But this alternative also fails because such a layer would have an Ekman-layer extension with an (uncorrectable) heat flux of order $E^{-1/4}$ at $z = \pm 1$.† The conclusion is that the solution of system (2.5a–f), or its equivalent (2.8a, b), must fulfil

$$\langle T_r \rangle = 0 \quad \text{at} \quad r = r_{i,o}. \tag{2.11}$$

There seems to be no difficulty to correct $T_r - \langle T_r \rangle$ at the vertical boundaries by constructing an $E^{1/3}$ layer solution of the kind discussed above. The reader is referred to Bark & Hultgren (1979) for details.

The remaining boundary condition is derived by considering the vertically averaged geostrophic radial volume flux $\langle u \rangle$. It is a straightforward matter to compute the radial volume flux $Q_E(\tau, r)$, say, in each of the Ekman layers by using (2.7). The result is

$$Q_E = \pm \frac{\pi}{\alpha^2} T_z(\tau, r, \pm 1). \tag{2.12}$$

† Regarding the viscous adjustment of the swirl velocity field, the role of the $E^{1/4}$ layer that appears in the homogeneous case ($M = 0+$) is in the present problem included in the equations for the interior motion (2.5). Thus, in the present case, the interior motion may be regarded as that of a degenerate $E^{1/4}$ layer.

It follows directly from $(2.5a, f)$ and the definition of the function $\rho_0(r)$ in (2.2) that

$$v = \tfrac{1}{2}(rT + \Xi_r). \tag{2.13}$$

If this equation is differentiated with respect to z at $r = r_{i,o}$ and use is made of $(2.5c)$ and the second of the boundary conditions in (2.9) it follows that $T_z = 0$ at $r = r_{i,o}$. Equation (2.12) then gives that $Q_E = 0$ at the vertical boundaries, which, in turn, implies that the vertically averaged volume flux of the geostrophic flow must also vanish at these boundaries. The remaining boundary condition is thus

$$\langle u \rangle = 0 \quad \text{at} \quad r = r_{i,o}. \tag{2.14}$$

This completes the mathematical formulation of the problem. The solution of the system of equations $(2.5a\text{–}f)$ with the initial condition (2.6) and the boundary conditions (2.9), (2.11) and (2.14) has to be computed by use of numerical methods. Results of such calculations are presented in the next section. Further analytic simplifications can be made for small values of α^2. This limit is considered in §4.

The asymptotic steady solution of the problem formulated above, which is a state of rigid rotation at constant temperature, is readily computed. In principle, there appear to be no difficulties in computing the modifications of the steady solution that would be caused by introducing gravity, which has been neglected in this work. Thus, in contrast to the case of a Boussinesq fluid, which as pointed out by Greenspan (1990, §1.4) cannot be in complete equilibrium in a state of rigid rotation, such an equilibrium is possible for a rotating isothermal gas. In both these cases, the isopycnic surfaces are paraboloids. For a Boussinesq fluid, the isothermal surfaces are also paraboloids. As shown by Greenspan (1990), this will always lead to a non-divergent diffusive heat flux and hence convection, albeit generally weak. For an isothermal gas, however, this situation does not occur and a stratification, which may vary in both the r- and z-directions, appears as the equilibrium state.

It may be worth pointing out that, in the limit of vanishing Mach number, the solution of the problem defined in this section *does not* reduce to the solution for spin-up of a homogeneous fluid in a container of finite size that was given by Greenspan & Howard (1963). Thus, in contrast to the case dealt with by Bark *et al.* (1978), the limits $E \to 0+$ and $M \to 0+$ are not commutative. The formal reason for this somewhat disturbing circumstance is that the scaling of the time variable and the dependent variables in terms of fractional powers of the Ekman number E used in this work is different from that in the corresponding problem for a homogeneous fluid. For example, in the latter case, spin-up is completed on the (dimensional) timescale $E^{-1/2}\Omega^{-1}$. On the timescale under consideration in this work, i.e. $E^{-1}\Omega^{-1}$, the spin-up of a homogeneous fluid would thus take place during a scaled non-dimensional time interval of zero length. Therefore, the solution of the present problem is singular for $M = 0$. Another closely related manifestation of this singular behaviour is the fact that, in the present case, the order of magnitude of the flow in the Ekman layers is $\epsilon E^{1/2}$ whereas, for the case of a homogeneous fluid, the corresponding order of magnitude is ϵ. Thus, the order of magnitude of the scaled non-dimensional Ekman-layer flow in the present case goes to infinity for vanishing Mach number. The mathematical structure of the solution of the present problem for small but finite values of M appears to be complicated and investigation of this matter is left for future work. The physical reasons for the aforementioned mathematical singularity are, however, obvious. For spin-up of a homogeneous fluid, the geostrophic motion is inviscid. In the present case, on the other hand, the motion outside the boundary layers is affected by diffusion, both viscous and thermal, to lowest order.

An issue that is closely related to the previous discussion is the absence of Stewartson $E^{1/4}$ layers in the present problem. It turns out that the approximate equations (2.5*a–e*) for conservation of momentum and heat express the same physics as the equations for the unsteady Stewartson $E^{1/4}$ layers that appear in a rapidly rotating gas during spin-up on the $E^{-1/2}\Omega^{-1}$ timescale in a container whose walls are thermally conducting, see equations (A 3*a–f*) in Bark *et al.* (1978). Thus, on the longer timescale $E^{-1}\Omega^{-1}$ considered in this work, the Stewartson $E^{1/4}$ layers have been given enough time to diffuse into the interior. These thickened boundary layers have thereby replaced the inviscid and non-diffusive flow that appears outside in the interior on the shorter timescale $E^{-1/2}\Omega^{-1}$. This effect was also observed by Matsuda & Hashimoto (1976) in a somewhat different parameter regime.

Some comments, albeit perhaps somewhat speculative, can be made on the expected nature of the motion in the limit $r_i \to 0+$, which is equivalent to removing the inner cylinder. The following arguments rest on the reasonable assumption that all dependent variables in the interior are independent of z for sufficiently small values of r. It is readily verified *a posteriori* that this assumption is self-consistent. To begin with, one notes that the gradient of the unperturbed density field $\rho_0(r)$ vanishes in the limit considered. Also, the net fluxes of heat and mass into the Stewartson $E^{1/3}$ layer on a cylindrical wall of infinitesimal radius are prescribed to be zero, see boundary conditions (2.11) and (2.14). This implies that the thermodynamic state of the gas in some small region near the axis of symmetry will change on the slow timescale $E^{-1}\Omega^{-1}$ only due to adiabatic expansion, which indicates that the spatial variation of the density and temperature fields will be modest. (For the density distribution for example, one would, for reasons of cylindrical symmetry, expect something like $\rho_0(\rho,\tau)) \approx A(\tau)+B(\tau)r^2+\dots$.) Thus, close to the axis of symmetry, the gas will spin up like a homogeneous fluid on the timescale $E^{-1/2}\Omega^{-1}$, which, when measured on the slow diffusive timescale, is instantaneous. It may be worth pointing out that this observation is consistent with the boundary condition (2.9) for the swirl velocity field for $r_i \to 0+$, which says that, in the neighbourhood of an inner cylinder of vanishing radius, the gas spins up immediately. Obviously, the rapid response of the gas near the axis of symmetry leads to serious difficulties not only in the present simplified problem but also in a numerical solution of the complete Navier–Stokes equations.

3. Numerical results

The mathematical problem defined in the previous section was solved numerically for different gases, container geometries and rates of rotation. The computational scheme is outlined in Appendix A. In order to keep the number of graphs at a reasonable level, attention will in what follows be restricted to a container geometry with $r_i = 0.25$ and $r_o = 0.75$. Results for UF_6 at room temperature with $\gamma = 1.067$ and $\sigma = 0.95$ will be discussed in some detail. The rate of rotation is chosen such that the Mach number $M = 2$, which, for UF_6 gives $\alpha^2 = 0.11$. The character of the motion for other values of γ and M will be discussed in qualitative terms at the end of this section.

Level curves for the swirl velocity v and the temperature T at three different instants of time are shown in figures 2(*a–c*) and 3(*a–c*), respectively. As all field quantities except w are symmetric with respect to the plane $z = 0$, only the upper half of the container is shown (w is antisymmetric with respect to $z = 0$). The evolution of the swirl velocity field show three very distinct features. First, apart from a boundary-layer-like region, whose thickness increases with time, outside the Ekman layers on the horizontal walls, the swirl velocity is practically independent of z. The motion has a

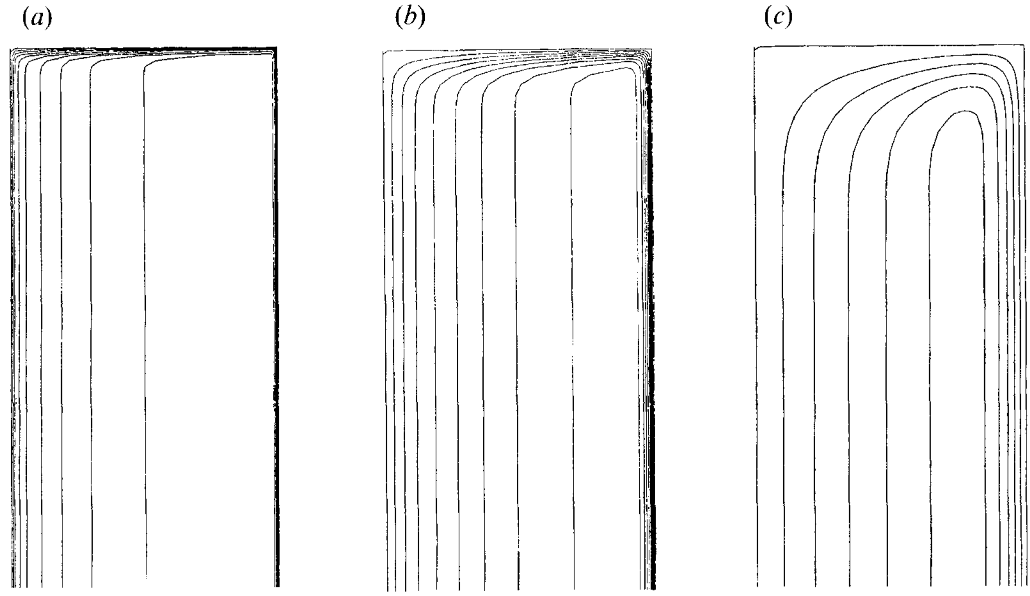


FIGURE 2(*a*–*c*). Level curves for the swirl velocity field v at different times for $M = 2$. In all graphs, $\Delta v = 0.1$ between level curves. (*a*) $\tau = 10^{-5}$, (*b*) $\tau = 10^{-4}$, (*c*) $\tau = 10^{-3}$.

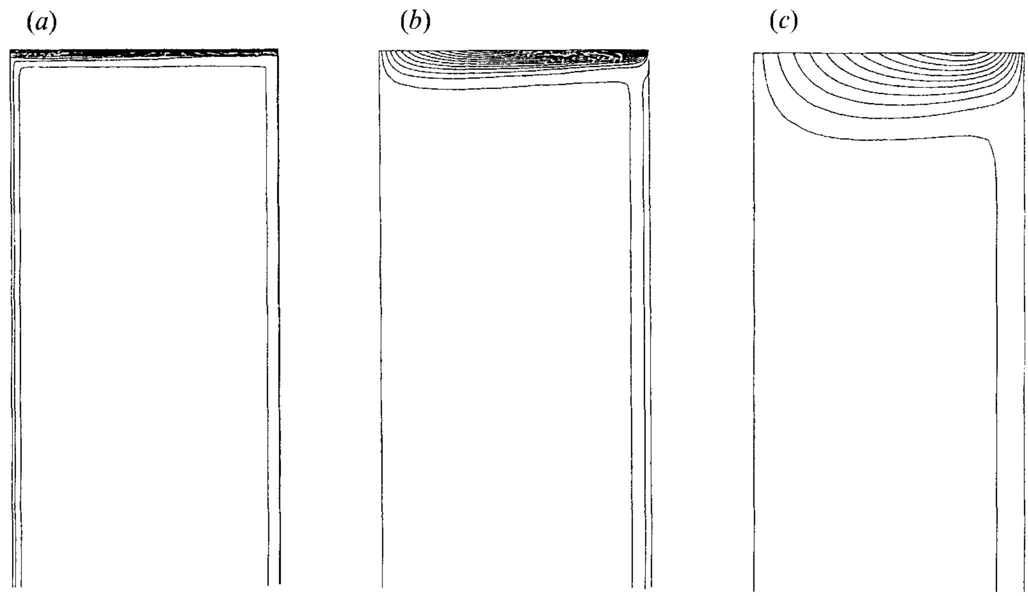


FIGURE 3(*a*–*c*). Level curves for the temperature field T at different times for $M = 2$. In all graphs, $\Delta T = 0.1$ between level curves. (*a*) $\tau = 10^{-5}$, (*b*) $\tau = 10^{-4}$, (*c*) $\tau = 10^{-3}$.

columnar character that is reminiscent of the spin-up of a homogeneous fluid. This indicates that the spin-up in the present case is controlled by some inviscid mechanism akin to Ekman suction, in spite of the fact that, as was shown in the previous section, effects of Ekman suction are no stronger than those due to diffusion. Secondly, the spin-up is unexpectedly fast. The swirl velocity field that is shown in figure 2(*c*) is, except close to the horizontal walls and close to the outer cylindrical wall, not very far from the asymptotic state of rigid rotation as early as for $\tau = 10^{-3}$. For $\tau = 10^{-2}$, one finds that the deviation from rigid rotation is of order 0.05. The third notable feature of the swirl velocity field is that effects of diffusion near the inner vertical wall are stronger than near the outer wall. This is simply a consequence of the strong radial variation of the basic density field. This variation increases strongly with the Mach number.

As can be inferred from figure 3(*a–c*), the nature of the evolution of the temperature field is also unexpected. The boundary-layer character of the temperature distribution outside the Ekman layers and the presence of a short timescale are again evident. Furthermore, outside the boundary-layer-like region and outside thin regions of (weak) variation near the vertical walls, the gas is practically isothermal.

For gases that are not very heavy, i.e. gases with $\gamma - 1 \sim 1$, numerical computations show that the boundary-layer character of the geostrophic motion, for a fixed value of the Mach number, is significantly less pronounced than for heavy gases. In the case of air, for which $\gamma - 1 \approx 0.4$, the aforementioned effects can still be discerned but are considerably weaker than for UF_6. If the value of γ is kept fixed and the Mach number is increased, the boundary-layer nature of the geostrophic flow again becomes less evident. These observations suggest that the character of the motion is determined by the value of the parameter α^2.

The result that, for Mach numbers of order unity, the main part of a heavy gas spins up inviscidly and isothermally on a very short timescale relative to the diffusion timescale will be investigated in some detail in the following two sections.

4. Approximate solution for a heavy gas

For a very heavy gas, the ratio γ of the specific heats at constant pressure and volume, respectively, is only slightly larger than unity. As $\alpha^2 \sim (\gamma - 1)$, the parameter α^2 is very small for such gases provided that M is not too large. In the present section, an approximate solution of (2.5*a–f*) for the geostrophic motion will be computed in the limit

$$\alpha^2 \to 0+, \quad \beta = \gamma\alpha^2/(\gamma-1) = \gamma\sigma M^2/(2r_o)^2 = O(1). \tag{4.1}$$

For small values of α^2, the results of the previous section show that the solution, outside the Ekman and Stewartson layers, is of boundary-layer character, with diffusive effects confined to regions close to the container walls. The nature of the approximate solution in the limit (4.1) can be inferred from the Ekman-suction formula (2.7). This formula shows that the Ekman-suction velocity becomes infinite in the limit (4.1). This somewhat surprising result is further discussed in §5. However, one may already at this stage infer that outside some neighbourhood of the boundaries, where viscous and thermal diffusion can be expected to be of importance, the gas will initially respond very rapidly in an inviscid and non-diffusive manner to a very strong Ekman suction. The fact that the Ekman-suction velocity is singular as $\alpha^2 \to 0+$ thus suggests that another (short) timescale is of relevance. A simple order-of-magnitude analysis of the equations of motion (2.5*a–f*) accounting for the order of magnitude of the Ekman-suction velocity as given by (2.7) shows that a meaningful perturbation problem results if time and the dependent variables outside anticipated regions of boundary-layer character near the walls, which will be analysed later, are scaled as follows:

$$\bar{\tau} = \tau/\alpha^4, \quad (u, v, w, p, \rho, T) = (\alpha^{-4}u_I, v_I, \alpha^{-4}w_I, p_I, \rho_I, \alpha^2 T_I). \tag{4.2}$$

Here quantities with the subscript I are assumed to be of order unity. Substitution of

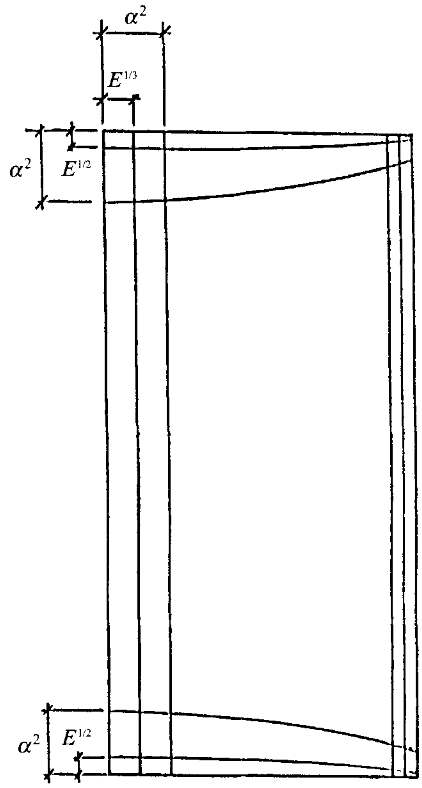


FIGURE 4. Boundary-layer structure for small values of α^2.

these scaled variables into (2.5*a–d*) gives the following lowest-order equations (with $\Xi = p/\rho_0$)

$$-2v_I + \Xi_{Ir} = 0, \quad v_{I\bar{\tau}} + 2u_I = 0, \quad \Xi_{Iz} = 0, \tag{4.3a–c}$$

$$\frac{4\beta}{\sigma}\Xi_{I\bar{\tau}} + \frac{1}{r\rho_0}(r\rho_0 u_I)_r + w_{Iz} = 0. \tag{4.3d}$$

These equations imply that the motion is not only geostrophic but also barotropic. It follows from (4.3*a–d*) that all dependent variables, except w_I and T_I, are independent of z; w_I is linear in z and the higher-order temperature field $\alpha^2 T_I$ does not have to be considered in order to compute the velocity and pressure fields to lowest order. The solution of (4.3*a–d*) can obviously not fulfil all boundary conditions at the walls. Boundary-layer solutions must thus be constructed. It should be pointed out that such boundary-layer solutions are local solutions of the system of equations (2.5*a–f*) for the motion of the geostrophic flow and are thus 'outer' solutions as seen from the Ekman and Stewartson $E^{1/3}$ layers. For consistency, one must thus require that the thicknesses of the boundary-layer parts of the geostrophic motion are larger than the thicknesses of the Ekman and Stewartson $E^{1/3}$ layers. It will turn out that this implies that the condition $E^{1/3} \ll \alpha^2 \ll 1$ has to be fulfilled. The boundary-layer structure of the flow is shown in figure 4.

To compute the boundary layer near the bottom and top walls of the container, a stretched coordinate and scaled variables are defined as follows

$$\zeta = (1 \pm z)/\alpha^2, \quad (u, v, w, p, \rho, T) = (\alpha^{-4}u_H, v_H, \alpha^{-2}w_H, 0, \rho_H, T_H), \tag{4.4}$$

where quantities denoted by the subscript H are assumed to be of order unity. The statement that the pressure is identically equal to zero follows from (2.5*c*) and implies that the gas in the boundary layer will behave as a Boussinesq fluid. From (2.5*a–f*) one finds the following equations to lowest order:

$$2\rho_0 v_H + r\rho_H = 0, \quad \rho_0 v_{H\bar{\tau}} + 2\rho_0 u_H = v_{H\zeta\zeta}, \tag{4.5a, b}$$

$$\rho_H\,\bar{\tau} + \frac{1}{r}(r\rho_0 u_H)_r \pm \rho_0 w_{H\zeta} = 0, \quad \sigma\rho_0 T_{H\bar{\tau}} = T_{H\zeta\zeta}, \quad \rho_H = -\rho_0 T_H. \tag{4.5c–e}$$

The solutions of (4.3*a–d*) and (4.5*a–e*) are separately required to fulfil the initial conditions (2.6). According to (2.7), (2.9) and the scaling rules (4.1) and (4.4) one must also prescribe

$$v_H + v_I = rH(\bar{\tau}), \quad w_I = \pm\frac{1}{4r\rho_0}T_{Hr\zeta}, \tag{4.6a, b}$$

for $z = -1$ and $z = 1$ respectively. $H(\bar{\tau})$ is the Heaviside step function. The solution for v_H in terms of the so far unknown function v_I is

$$v_H = \operatorname{erfc}\,[(\rho_0\sigma/4\bar{\tau})^{1/2}\zeta] - \int_0^{\bar{\tau}} \operatorname{erfc}\,[(\rho_0\sigma/4\bar{\tau}')^{1/2}\zeta]\, v_{I\bar{\tau}}(r, \bar{\tau}-\bar{\tau}')\,\mathrm{d}\bar{\tau}'. \tag{4.7}$$

The interaction between the motion in the horizontal boundary layers and that in the central parts of the container is now clear. Owing to viscous diffusion, the increased rotation rate of the container walls is immediately felt by the gas in the neighbourhood of the horizontal boundaries. This motion is accompanied by a temperature field, which, according to (4.5*a, e*), is similar to the ‘thermal wind’ encountered in geophysical fluid dynamics, see e.g. Holton (1979). This temperature field is not compatible with the boundary condition at the adiabatic walls and is corrected by (divergent) Ekman layers. As $T_{H\zeta}$ is of order α^{-2}, the Ekman suction velocity will, according to (2.7), be of order α^{-4}, i.e. of the same order of magnitude as the axial velocity $\alpha^{-4}w_I$ of the inviscid and adiabatic flow but larger than the axial velocity $\alpha^{-2}w_H$ in the horizontal boundary layers. Gas from the interior is thus sucked into the Ekman layers right through the horizontal boundary layers.

An equation for v_i can be derived as follows: u_I and $\Xi_{Ir\bar{\tau}}$ are expressed in terms of v_I by using (4.3*a, b*) and are inserted into the derivative with respect r of the equation of continuity (4.3*d*). The resulting equation is thereafter integrated from $z = -1$ to $z = 1$. One then obtains an equation that contains partial derivatives of v_I and a linear combination of $w_{Ir}(r, \pm 1, \bar{\tau})$. These latter quantities can be expressed in terms of v_I by using (4.5*a, e*), (4.6*b*) and (4.7). After a little calculation, this leads to the following singular partial integro-differential equation for $v_I(r, \bar{\tau})$:

$$\left[\frac{1}{r\rho_0}(r\rho_0 v_{I\bar{\tau}})_r\right]_r - \frac{16\beta}{\sigma}v_{I\bar{\tau}} + \left(\frac{\sigma}{\pi}\right)^{1/2}\left\{\frac{1}{r\rho_0}\left[\rho_0^{1/2}\left(\frac{1}{r}\int_0^{\bar{\tau}}\frac{v_{I\bar{\tau}}(\bar{\tau}')\,\mathrm{d}\bar{\tau}'}{(\bar{\tau}-\bar{\tau}')^{1/2}} - \frac{1}{\bar{\tau}^{1/2}}\right)\right]_r\right\}_r = 0, \tag{4.8}$$

where β is defined by (4.1). In addition to the initial condition $v_I(r, 0) = 0$, boundary conditions at $r = r_{i,o}$ are needed to determine the solution of this equation. These boundary conditions can be derived by considering the structure of the lowest-order mathematical problems for the motion in the boundary layer (outside the Stewartson layers) at $r = r_{i,0}$ and the motion in the corner regions in the neighbourhood of $r = r_{i,o}$, $z = \pm 1$. The calculations are carried out in Appendix B and the result is

$$v_I(r_{i,o}, \bar{\tau}) = r_{i,o}\{1 - \exp\,[\lambda^2(r_{i,o})\,\pi\bar{\tau}]\,\operatorname{erfc}\,[\lambda(r_{i,o})\,(\pi\bar{\tau})^{1/2}]\}, \tag{4.9}$$

$$\lambda(r) = \left(\frac{\sigma}{\pi\rho_0(r)}\right)^{1/2}\frac{1}{r^2}.$$

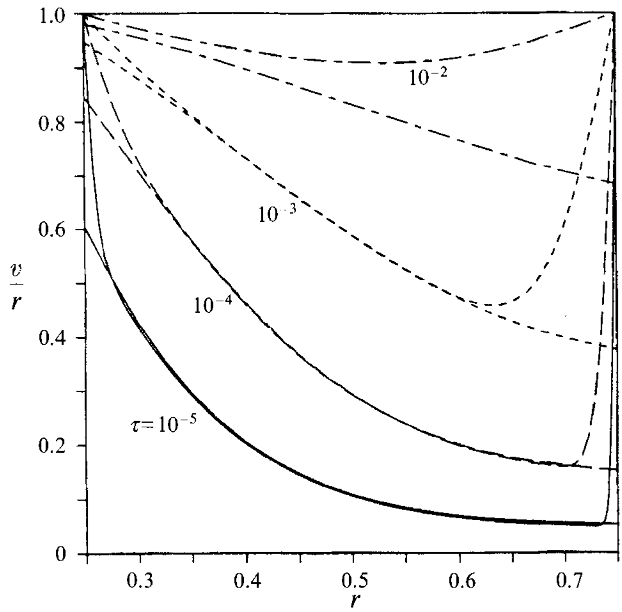


FIGURE 5. Comparison between different approximate solutions for the swirl velocity field v for $M = 2$ at $z = 0$. Solutions of equations (2.5a–f) which include diffusive effects and equation (4.8) for the non-diffusive interior. The boundary-layer-like behaviour of the diffusive solution should be noted. The values of τ are 10^{n-6}, $n = 1-4$. The corresponding values for $\bar{\tau}$ are $7.81 \times 10^{n-5}$, $n = 1-4$.

The numerical solution of the simplified problem for $v_I(r, \tau)$ defined by (4.8), the initial condition $v_I(r, 0) = 0$ and the boundary condition (4.9) have been compared with the numerical solution of the simplified but still more realistic model problem defined in §2. A comparison between the two solutions at $z = 0$ is shown in figure 5 for the same parameter settings as in figures 2 and 3.

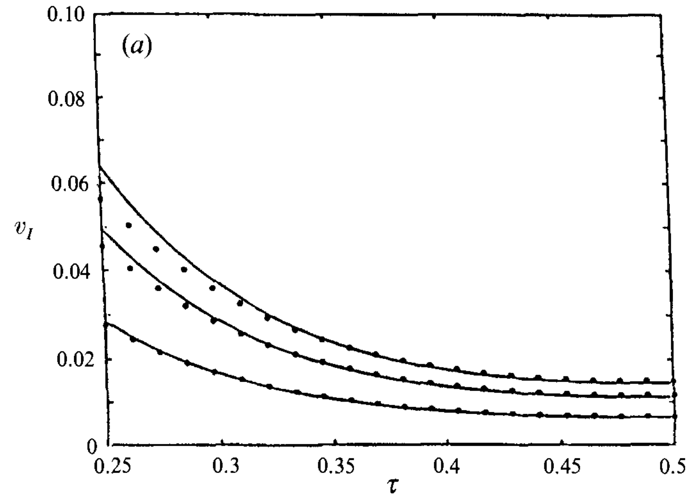
Figure 5 shows that, outside the radial boundary layers present in the viscous model, the agreement between the two solutions is excellent for small values of τ ($\bar{\tau}$ smaller than 1). The motion in these regions is briefly discussed in Appendix B. For $\tau \sim 10^{-2}$, the effects of diffusion are felt everywhere. Thus, even though the initial phase of the spin-up is distinctly inviscid and isothermal, the final adjustment to rigid rotation will involve effects of diffusion. Stated somewhat more precisely, one finds that the approach to rigid rotation outside the vertical and horizontal boundary layers is completely inviscid and isothermal in the limit $\gamma \rightarrow 1+0$ with M fixed For a small but finite value of $\gamma - 1$, diffusive effects will take part in the later stage of the process.

Solutions of the problem defined by (4.8) and (4.9) for small and large values of $\bar{\tau}$ are computed in Appendix C. These solutions are of the form

$$v_I(r, \bar{\tau}) = \rho_0^{-1/2}\bar{v}(r)\,\bar{\tau}^{1/2} + O(\bar{\tau}), \quad \bar{\tau} \ll 1, \tag{4.10}$$

$$v_I(r, \bar{\tau}) = r + r^{-1}\rho_0^{1/2}\tilde{v}(r)\,\bar{\tau}^{-1/2} + O(\bar{\tau}^{-1}), \quad \bar{\tau} \gg 1. \tag{4.11}$$

An ordinary differential equation for the function $\bar{v}(r)$ and an analytical expression the function $\tilde{v}(r)$ are given in Appendix C. It can be shown that the result $v_I = O(\bar{\tau}^{1/2})$ for small values of $\bar{\tau}$ holds for any value of α. The algebraic details are complicated and are omitted here. It follows from (4.10) and (4.3a, b, d) that u_I and w_I are $O(\bar{\tau}^{1/2})$ for small values of $\bar{\tau}$, which is a manifestation of the non-uniform validity of the solution as $\bar{\tau}$ approaches zero. The solution (4.11) shows that the approach to the steady state in the limit $\gamma \rightarrow 1+$, M fixed, is algebraic. A comparison of the small- and large-time



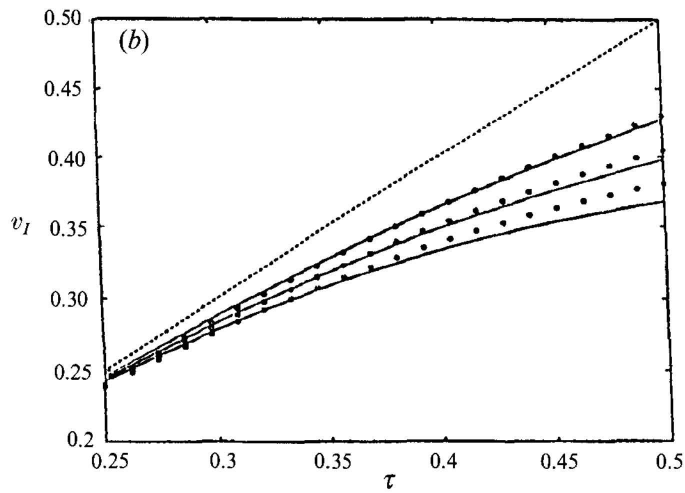


FIGURE 6. Comparison between the approximate solutions (solid curve) and the numerical solution of equation (4.8) (dots) for $M = 2$. (*a*) Approximate solution (4.10); the values of $\bar{\tau}$ are 10^{-5}, 3×10^{-5} and 5×10^{-5}. (*b*) Approximate solution (4.11); the values of $\bar{\tau}$ are 0.3, 0.5 and 1.

approximations with the solution of (4.8) in figures 6(*a*) and 6(*b*) shows that the simplified expressions capture well the initial and final phases of the inviscid spin-up process. The large-time behaviour however, may be of more academic interest as viscous effects dominate the later stages of the spin-up for small but finite $\gamma-1$ as was demonstrated previously.

5. Discussion

The role of the Ekman layers in the spin-up mechanism investigated in the previous sections may bè worth some comments. In the case of a homogeneous fluid, as well as in the somewhat similar case of a rapidly rotating gas in a thermally conducting container, the Ekman layers exert active control on the swirling motion in the interior (see e.g. Greenspan & Howard 1963 and Bark *et al.* 1978, respectively). However, in the case of a heavy gas in a container with thermally insulating walls, the effects of the Ekman-suction mechanism is a little more complicated. In turns out, though, that some insight, including a physical argument for the reduced strength of the flow in the Ekman layers, can be obtained from direct inspection of the equations for the Ekman layers in the form of Bark *et al.* (1978). In these layers, which are quasi-steady on the diffusive timescale, the dependent variables and the boundary-layer coordinate η are scaled as follows:

$$(u, v, w, p, \rho, T) = E^{\Gamma}(u_E, v_E, E^{1/2} w_E, Ep_E, \rho_E, T_E), \quad \eta = (1 \pm z)/E^{1/2}, \tag{5.1}$$

where quantities denoted by the subscript E are assumed to be of order unity and the exponent Γ in the scaling factor for the motion as a whole in the boundary layer is to be determined. Integration of the Ekman-layer version of the energy equation with respect to η between 0 and ∞ gives, for example equation (13d) in Bark *et al.* (1978),

$$T_{E\eta}(r, 0, t) = 4\alpha^2 r \rho_0 \int_0^{\infty} u_E(r, \eta, t)\, \mathrm{d}\eta. \tag{5.2}$$

This relation quantifies the local balance (at each value of r) between vertical diffusion of heat to/from the boundary and the net rate of positive/negative production of heat that is caused by compression work per unit time on gas particles that are moving radially in the stratified background density field ρ_0. Owing to the *local* homogeneity of the Ekman layer in the direction of $\boldsymbol{e}_r$, all heat produced by compression work at each radial location in the layer must thus be removed by diffusion and, so to speak, be taken over by the boundary-layer-like part of the geostrophic motion as expressed by the boundary condition (2.3) for the temperature field

$$E^{\Gamma - 1/2} T_{E\eta} + \alpha^{-2} T_{H\zeta} + \ldots = 0, \quad \eta = \zeta = 0. \tag{5.3}$$

As a consequence of the shape of the velocity distribution in the Ekman layer, the net rate of production of heat due to compression in the boundary layer, i.e. the right-hand side of (5.2), is always non-zero. This is fairly obvious on intuitive grounds and can readily be shown mathematically. Consequently, if the strength of the motion in the Ekman layer is proportional to the Rossby number ϵ, i.e. $\Gamma = 0$ in (5.1), there is a non-zero diffusive heat flux of order $\epsilon E^{-1/2}$ at the horizontal boundaries, which is much too large to be removed by the $O(\epsilon\alpha^{-2})$ heat flux associated with the boundary-layer-like part of the interior motion and is therefore not compatible with the boundary condition on the adiabatic walls. However, if the Ekman layer is weak in the sense that $\Gamma = \frac{1}{2}$, the aforementioned two heat fluxes can cancel at the horizontal boundaries in order to fulfil the adiabatic constraint to lowest order. These arguments are, in several respects, quite similar to those given by Matsuda & Hashimoto (1976) and Matsuda, Hashimoto & Takeda (1976), who considered similar problems but in different parameter régimes. The result $\Gamma = \frac{1}{2}$ was derived by Bark & Hultgren (1979) on purely formal grounds.

The singular behaviour of the Ekman suction for small values of α^2 and the scaling relations (4.1) and (4.4) that were formally postulated in the previous section can also be given a physical interpretation by considering (5.2). This equation implies that a diffusive heat flux $T_{E\eta}$ of order unity in the Ekman layer is set up by a large radial velocity u_E of order α^{-2}. This equation implies that the diffusive heat flux in the Ekman layer, i.e. $T_{E\eta}$, which has to be of order unity to annihilate the heat flux from the geostrophic motion at the insulated boundaries, forces a large radial velocity $\sim \alpha^{-2}$ in the Ekman layer. This is a consequence of the fact that a heavy gas has many internal degrees of freedom, which means that a large amount of thermal energy has to be supplied to such a gas in order to obtain a given change of temperature. (The temperature is a measure of the translational modes only.) In the present case, that power is supplied by compression (or expansion) due to rapid radial motion in the pressure field of the basic state. The large radial velocity leads, of course, to a large axial velocity, which is accounted for by the factor α^{-2} in the Ekman-suction formula (2.7). The authors owe this argument to Professor Takeo Sakurai.

The fact that the boundary-layer character of the geostrophic motion, as was discussed at the end of the previous section, disappears for increasing values of the Mach number for a small but finite value of $\gamma-1$, can also be explained by (5.2). If the Mach number is increased, the radial variation of the basic density field is increased. Thus, the radial velocity in the Ekman layer that is needed to provide a given rate of compression work is reduced. A reduced radial velocity in the Ekman layer will, owing to continuity, reduce the Ekman suction, which, in turn, means that the boundary-layer character of the geostrophic motion becomes weaker.

Formula (2.7) was derived by Bark & Hultgren (1979) under the assumption that the gradient of the interior temperature field at the horizontal boundaries is of order unity. For small values of α^2, though, that gradient is of order α^{-2}, see (5.3), owing to the boundary-layer character of the temperature field outside the Ekman layers. This means that the axial velocity in the Ekman layer, which is communicated to the interior, is of order α^{-4}. The strong Ekman-layer suction, in turn, sets up a meridional circulation of strength α^{-4} in the interior of the container as was deduced formally in the previous section, see the scaling given by (4.2). For heavy gases, the spin-up timescale is thus reduced from $E^{-1}\Omega^{-1}$ to $(\gamma-1)^2E^{-1}\Omega^{-1}$.

6. Conclusions

It has been shown that a rapidly rotating gas in a container, whose walls are adiabatic, adjusts to a suddenly increased rotation rate of the container on the diffusive timescale $E\Omega^{-1}$, in contrast to the $E\Omega^{-1/2}$ timescale that is the case for a homogeneous fluid. The reason is that the motion in the Ekman layers is a factor $E^{1/2}$ weaker than in the homogeneous fluid case, which means that the timescale for vortex stretching due to the Ekman-suction mechanism is of the same order of magnitude as that for viscous and thermal diffusion. The role of the weak Ekman layers is to cancel the heat flux associated with the geostrophic motion at the horizontal walls. In the homogeneous fluid case, there are two overlapping boundary layers at the vertical walls, the Stewartson $E^{1/4}$ and $E^{1/3}$ layers, respectively. These layers adjust the inviscid geostrophic motion in the interior to the no-slip condition at the vertical walls. In the present case, only the $E^{1/3}$ layer appears. The role of the $E^{1/4}$ layer has been taken over by the interior motion, which is partially controlled by viscous and diffusive effects.

For heavy gases, which are characterized by small values of $(\gamma-1)$, that are rotating at angular velocities such that both the parameter combinations $M^2(\gamma-1)$ and $E^{1/3}(\gamma-1)^{-1}$ are small, the spin-up process takes place on a reduced timescale $(\gamma-1)^2E^{-1}\Omega^{-1}$. The primary reason for this perhaps somewhat unexpected result is that, for small values of $(\gamma-1)$, the Ekman-suction velocity is proportional to $(\gamma-1)^{-1}$ provided that the temperature gradient of the interior is of order unity at the horizontal walls. This indicates a shorter timescale for spin-up. However, a shorter timescale means that, outside the Ekman and Stewartson layers, the effects of viscous and thermal diffusion are felt by the gas only in the neighbourhood of the boundaries whereas the Ekman suction has a global effect. One finds that the motion outside the

Ekman and Stewartson layers splits into a non-diffusive and inviscid interior region, in which the motion of the gas is barotropic and geostrophic, and boundary layers of thickness proportional to $(\gamma-1)$ at the horizontal and vertical walls. The weak Ekman layers must thus correct a temperature gradient of order $(\gamma-1)^{-1}$, which results in an Ekman suction velocity of order $(\gamma-1)^{-2}$ This suction velocity, in turn, gives a (dimensional) timescale for spin-up of $(\gamma-1)^2E^{-1}\Omega^{-1}$.

The authors owe the solution (B 14) in Appendix B to Professor emeritus Bengt Joel Andersson of the Royal Institute of Technology. Many valuable points of view on the present work, especially the contents of §5, were given by Professor Takeo Sakurai of Kyoto University and Professor Harvey P. Greenspan of Massachusetts Institute of Technology. Professor Nobumasa Sugimoto of Osaka University gave several useful suggestions on the manuscript.

Appendix A. Numerical method

The system of equations (2.5) was integrated numerically, using initial conditions (2.6) and boundary conditions (2.7), (2.9), (2.11) and (2.14). Prior to the discretization of the equations, they are reformulated in terms of

$$U = ur, \quad V = v/r, \quad W = wr, \quad \Theta = T-2v/r, \quad \Pi = p/\rho_0, \tag{A 1}$$

where Φ, Π depend on r and τ only.

In order to allow a non-equidistant spacing in the container the r- and z-coordinates are mapped to a computational $\xi(r)$, $\zeta(z)$ domain covering the region $(0,0)$ to $(1,1)$. The equations are discretized in the ξ,ζ coordinates using central second-order-accurate difference formulae. In order to enable a compact second-order space discretization of the equations the variables are stored on a staggered grid. The space discretization is, including dummy points to take care of the boundary conditions

$$\xi_i = ih_r, \quad h_r = 1/N_r, \quad \zeta_j = jh_z, \quad h_z = 1/N_z, \tag{A 2}$$

$$U_{i,j} = U(\xi_i, \zeta_{j-1/2}, \tau), \quad 0 \leqslant i \leqslant N_r, \quad 1 \leqslant j \leqslant N_z, \tag{A 3a}$$

$$V_{i,j} = V(\xi_{i-1/2}, \zeta_{j-1/2}, \tau), \quad 0 \leqslant i \leqslant N_r+1, \quad 0 \leqslant j \leqslant N_z+1, \tag{A 3b}$$

$$W_{i,j} = W(\xi_{i-1/2}, \zeta_j, \tau), \quad 1 \leqslant i \leqslant N_r, \quad 0 \leqslant j \leqslant N_z, \tag{A 3c}$$

$$\Pi_i = \Pi(\xi_{i-1/2}, \tau), \quad 0 \leqslant i \leqslant N_r+1, \tag{A 3d}$$

$$\Theta_i = \Theta(\xi_i, \tau), \quad -1 \leqslant i \leqslant N_r+1. \tag{A 3e}$$

The time differencing is done using Crank–Nicholson discretization, resulting in an implicit second-order scheme. The stability characteristics of this method are favourable in cases like the ones calculated below with a very large span in timescales over the computational domain. The variables are also stored at staggered time levels

$$\tau^k = k\,\Delta\tau, \tag{A 4}$$

$$U^k_{i,j} = U_{i,j}(\tau^{k-1/2}), \quad V^k_{i,j} = V_{i,j}(\tau^k), \quad W^k_{i,j} = W_{i,j}(\tau^{k-1/2}), \tag{A 5}$$

$$\Pi^k_i = \Pi_i(\tau^k), \quad \Theta^k_{i,j} = \Theta_{i,j}(\tau^k), \tag{A 6}$$

reflecting the lack of initial conditions and time derivatives for U, W in the equations. The results shown below were obtained on grids with $N_r = 40$, $N_z = 20$. The grid-point distribution is determined from the expected variation of the solution obtained from the asymptotic results of §4. Test calculations on grids with twice the resolution gave virtually the same results.

Appendix B. Derivation of boundary conditions for equation (4.8)

For the derivation of the boundary conditions for v_I at $r = r_{i,o}$, the vertical boundary layers as well as the corner regions must be considered. A consistent perturbation for the boundary-layer problem at $r = r_i$ can be formulated in terms of the following stretched coordinate and scaled dependent variables:

$$\xi = (r-r_i)/\alpha^2, \quad (u, v, w, p, \rho, T) = (\alpha^{-4}u_V, v_V, \alpha^{-6}w_V, \alpha^2 p_V, \alpha^2\rho_V, \alpha^4 T_v). \qquad \text{(B 1)}$$

Variables denoted by subscript V are assumed to be of order unity. (The layer at $r = r_o$ is, of course, computed in exactly the same way.) Substitution of (B 1) in (2.5*a–e*) gives the following system of equations:

$$2v_V + \Xi_{V\xi} = 0, \quad \rho_0(r_i)(v_{V\bar{\tau}} + 2u_V) = v_{V\xi\xi}, \quad \Xi_{Vz} = 0, \quad u_{V\xi} + w_{Vz} = 0. \qquad \text{(B 2}a\text{–}d)$$

Incidentally, these equations are of exactly the same form as those for the unsteady Stewartson $E^{1/4}$ layer (see Greenspan & Howard 1963). However, the solution in the present case is considerably more complicated owing to the coupling to the solution in the corner regions. It follows directly from (B 2*a–d*) that all dependent variables except w_V are independent of z; w_V is linear in z. The initial and boundary conditions for the system (B 2*a–d*) are

$$v_V(\xi, 0) = 0, \quad v_V(0, \bar{\tau}) + v_I(r_i, \bar{\tau}) = r_i H(\bar{\tau}), \quad v_V(\infty, \bar{\tau}) = 0. \qquad \text{(B 3)}$$

The boundary condition for v_I can be obtained without the solution of the system of equations (B 2*a–d*). As will be shown below, only some general properties of the solution are needed. A solution of (B 2*a–d*) is given at the end of this Appendix.

In the corner regions, the stretched variables ζ and ξ, which are defined by (4.4) and (B 1), are both of order unity and the dependent variables are scaled as

$$(u, v, w, p, \rho, T) = (\alpha^{-4}u_C, v_C, \alpha^{-4}w_C, 0, \rho_C, T_C). \qquad \text{(B 4)}$$

For notational simplicity, the necessary algebraic details will be given only for the corner region $r \approx r_i, z \approx -1$. One finds the following lowest-order equations:

$$-2\rho_0(r_i)v_C = r_i\rho_C, \quad \rho_0(r_i)(v_{C\bar{\tau}} + 2u_C) = v_{C\xi\xi} + v_{C\zeta\zeta}, \qquad \text{(B 5}a, b)$$

$$u_{C\xi} + w_{C\zeta} = 0, \quad \sigma\rho_0(r_i)T_{C\bar{\tau}} = T_{C\xi\xi} + T_{C\zeta\zeta}, \quad \rho_0(r_i)T_C + \rho_C = 0. \qquad \text{(B 5}c\text{–}e)$$

The solution of this system is required to satisfy the following initial and boundary conditions:

$$\left.\begin{aligned} v_C(\xi, \zeta, 0) = 0, \quad v_C(0, \zeta, \bar{\tau}) + v_H(r_i, \zeta, \bar{\tau}) = r_i H(\bar{\tau}), \quad v_C(\xi, 0, \bar{\tau}) + v_V(\xi, \bar{\tau}) = r_i H(\bar{\tau}), \\ v_C(\xi, \infty, \bar{\tau}) = 0, \quad v_C(\infty, \zeta, \bar{\tau}) = 0. \end{aligned}\right\} \qquad \text{(B 6)}$$

It should be noted that the axial velocity in the corner region is of higher order than that in the vertical layer. The axial mass flux in the vertical layer is thus supplied directly from the Ekman layer at the bottom of the corner region, i.e.

$$w_V = \frac{1}{4r_i\rho_0(r_i)}T_{C\xi\zeta}, \quad z = -1. \qquad \text{(B 7)}$$

The boundary condition for v_I is derived from the lowest-order approximation of (2.14), which reads

$$u_I + u_V = 0, \quad r = r_i. \qquad \text{(B 8)}$$

u_I is simply related to v_I by (4.3*b*). One may express u_V in terms of v_V from (B 2*d*), which gives

$$u_V(\xi, \bar{\tau}) = -\int_{\xi}^{\infty} w_V(\xi', -1, \bar{\tau})\, \mathrm{d}\xi', \tag{B 9}$$

where the linear dependence of w_V on z has been used. From (B 7) and (B 9) one then obtains

$$u_V = \frac{1}{4r_i \rho_0(r_i)} T_{C\xi}(\xi, 0, \bar{\tau}). \tag{B 10}$$

According to (2.9) and (2.13), it follows that the derivative with respect to z of the temperature field $T = T_I + T_V + T_H + T_C$ at $r = r_i$ is zero. To lowest order, this implies that

$$T_{H\zeta}(r_i, \zeta, \bar{\tau}) + T_{C\xi}(0, \zeta, \bar{\tau}) = 0. \tag{B 11}$$

$T_{H\zeta}$ can be expressed in terms of v_I by using (4.5*a*, *e*) and (4.7). Combination of the formula so obtained and (B 8) (B 10) and (B 11) leads to the following boundary conditions for v_I:

$$v_{I\bar{\tau}}(r_i, \bar{\tau}) + \left(\frac{\sigma}{\pi \rho_0(r_i)}\right)^{1/2} \frac{1}{r_i}\left[\int_0^{\bar{\tau}} \frac{v_{It}(r_i, \bar{\tau}')\, \mathrm{d}\bar{\tau}'}{r_i(\bar{\tau} - \bar{\tau}')^{1/2}} - \frac{1}{\bar{\tau}'^{1/2}}\right] = 0. \tag{B 12}$$

The solution of this integral equation is

$$v_I(r_i, \bar{\tau}) = r_i\{1 - \exp[\lambda^2(r_i)\pi\bar{\tau}]\, \mathrm{erfc}\,[\lambda(r_i)(\pi\bar{\tau})^{1/2}]\}, \quad \lambda(r) = \left(\frac{\sigma}{\pi\rho_0(r)}\right)^{1/2} \frac{1}{r^2}, \tag{B 13}$$

which is the boundary condition for (4.8).

The solution of the coupled system (B 2*a–d*) and (B 5*a–e*) with the boundary conditions (B 3) and (B 6) is, in the general case, rather complicated. For the special case $\sigma = 1$, which is close to the measured value $\sigma = 0.95$ for UF_6 at room temperature, a reasonably simple solution for v_V can be computed. In terms of the rescaled time variable $\bar{\tau}' = \rho_0(r_i)\bar{\tau}$, where the prime henceforth is dropped, and the constant $a = 1/(2r_i^2)$ one finds the following solution:

$$\begin{aligned} v_V = {} & \frac{\xi}{2\pi^{1/2}} \int_0^{\bar{\tau}} [r_i H(\tau - \bar{\tau}') - v_I(r_i, \bar{\tau} - \bar{\tau}')]\, \bar{\tau}'^{-3/2}\, \mathrm{erfc}\,(a\bar{\tau}'^{1/2}) \exp\{a^2\bar{\tau}' - \xi^2/4\bar{\tau}'\}\, \mathrm{d}\bar{\tau}' \\ & - \frac{a}{2\pi} \int_0^{\bar{\tau}} v_H(\bar{\tau} - \bar{\tau}') \int_0^{\bar{\tau}'} (\bar{\tau}' - \bar{\tau}'')^{-3/2} \exp(a^2\bar{\tau}'')\, \mathrm{erfc}\,(a\bar{\tau}''^{1/2}) \\ & \times \int_0^{\xi} [\bar{\tau}''^{-1/2} \exp(-\xi'^2/4\bar{\tau}'') - \bar{\tau}'^{-1/2} \exp(-\xi'^2/4\bar{\tau}')]\, \mathrm{d}\xi'\, \mathrm{d}\bar{\tau}''\, \mathrm{d}\bar{\tau}'. \end{aligned} \tag{B 14}$$

The details of the derivation of (B 14) are somewhat complicated and are not reproduced here. The manipulations are available on request from the authors.

Appendix C. Approximate solution of equation (4.8) for small and large values of τ

For small values of $\bar{\tau}$, the boundary condition (4.9) can be written

$$v_I(r_{i,o}, \bar{\tau}) = 2r_{i,o}\lambda(r_{i,o})\bar{\tau}^{1/2} + O(\bar{\tau}), \quad \bar{\tau} \ll 1, \tag{C 1}$$

which suggests that an approximate solution of (4.8) is of the form

$$v_I(r,\bar{\tau}) = \rho_0^{-1/2}\bar{v}(r)\,\bar{\tau}^{1/2}+O(\bar{\tau}), \quad \bar{\tau} \ll 1. \tag{C 5}$$

Substitution of this ansatz into (4.8) and (B 1) leads to the following problem for the function $\bar{v}(r)$:

$$\bar{v}''+\frac{1}{r}\bar{v}'-\left(\frac{16\beta}{\sigma}+\frac{4\beta^2r^2}{\sigma^2}+\frac{1}{r^2}\right)\bar{v} = -\frac{8\beta^2 r}{(\pi\sigma)^{1/2}}, \tag{C 3}$$

$$\bar{v}(r_{i,o}) = -\frac{2(\sigma)^{1/2}}{\pi^{1/2}r_{i,o}}. \tag{C 4}$$

It is a straightforward matter to solve this problem by numerical methods.

For large values of $\bar{\tau}$, the boundary condition (4.9) has the approximate representation

$$v_I(r_{i,o},\bar{\tau}) = r_{i,o}\left[1-\frac{1}{\pi\lambda(r_{i,o})\,\bar{\tau}^{1/2}}\right], \quad \bar{\tau} \gg 1, \tag{C 5}$$

which indicates that the solution of (4.8) is of the form

$$v_I(r,\bar{\tau}) = r+r^{-1}\rho_0^{1/2}\,\tilde{v}(r)\,\bar{\tau}^{-1/2}+O(\bar{\tau}^{-1}), \quad \bar{\tau} \gg 1. \tag{C 6}$$

An equation for $\tilde{v}(r)$ can be derived from the Laplace transform of (4.8). The transformed equation depends parametrically on the square root of the transform variable s and it is thus reasonable to assume that the Laplace transform of $\tilde{v}(r)$ can be expanded in powers of $s^{1/2}$ for small values of s. Under the further assumption that the Laplace transform of $\tilde{v}(r)$ is an analytic function of s except at $s=0$ and a branch cut along the negative real axis, the power series in $s^{1/2}$ can be inverted term by term (cf. Carrier, Krook & Pearson 1966, p. 356), which leads to the ansatz (C 6) and the following equation for $\tilde{v}(r)$:

$$\left[\frac{\tilde{v}'}{r\rho_0}\right]' = \frac{8br}{\pi^{1/2}\sigma^{3/2}}, \tag{C 7}$$

$$\tilde{v}(r_{i,o}) = \frac{r_{i,o}\,\rho_0(r_{i,o})}{\pi^{1/2}\sigma^{3/2}}. \tag{C 8}$$

This problem has the solution

$$\tilde{v}(r) = \frac{1}{\pi^{1/2}\sigma^{3/2}}(r^2\rho_0(r)+A\rho_0(r)+B), \tag{C 9}$$

with the constants A and B given by

$$A = -2\frac{r_o^2\,\rho_0(r_o)-r_i^2\,\rho_0(r_i)}{\rho_0(r_o)-\rho_0(r_i)}, \quad B = 2\frac{\rho_0(r_o)\,\rho_0(r_i)\,(r_o^2-r_i^2)}{\rho_0(r_o)-\rho_0(r_i)}. \tag{C 10}$$

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